MATHEMATICAL FORECASTING USING THE BOX-JENKINS METHODOLOGY

TECHNICAL BRIEFING

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Note: The document *The Box-Jenkins Forecasting Technique*, posted at http://www.foundationwebsite.org/BoxJenkins.htm , presents a nontechnical description of the Box-Jenkins methodology. For a technical description of the Box-Jenkins approach, see the document, *TIMES Box-Jenkins Forecasting System*, posted at http://www.foundationwebsite.org/TIMESVol1TechnicalBackground.htm . A computer program that can be used to develop a broad class of Box-Jenkins models is posted at the Foundation website, http://www.foundationwebsite.org (6 February 2009).

BRIEFING ROAD MAP

- 1. MATHEMATICAL FORECASTING CONCEPTS (20-30 MINUTES)
- 2. TECHNICAL INTRODUCTION TO THE BOX-JENKINS METHODOLOGY (20-30)
- 3. FORECASTING ACCURACY COMPARISONS (5-10)
- 4. MODEL BUILDING WITH THE BOX-JENKINS METHODOLOGY (40-60)
- 5. APPLICATION TO ECONOMETRIC AND CONTROL PROBLEMS (10-15)

1. MATHEMATICAL FORECASTING CONCEPTS

MATHEMATICAL FORECASTING METHODOLOGY (“FORECASTER”)

BASED ON A MATHEMATICAL MODEL OF THE PROCESS

TWO APPROACHES

- MODEL FITTING (INTUITIVE, HEURISTIC)
- MODEL BUILDING (THEORETICAL FOUNDATION)

HEURISTIC FORECASTERS

A PARTICULAR MODEL IS FITTED TO DATA

EXAMPLES:

- MOVING AVERAGE
- EXPONENTIAL SMOOTHING
- TRENDS, CURVES, HARMONICS, PATTERNS

GOOD FIT → GOOD FORECAST?

GOOD MODEL → GOOD FORECAST

NEED TO BUILD A GOOD MODEL

MODEL BUILDING

CHOOSE A COMPREHENSIVE CLASS OF MODELS

- IDENTIFICATION
- FITTING
- DIAGNOSTIC CHECKING

REPEAT UNTIL ADEQUATE MODEL CONSTRUCTED

CLASSES OF MODELS (FOR PREDICTION AND CONTROL)

ECONOMETRIC MODELS

- DYNAMIC CAUSAL MODEL (MANY VARIABLES)
- ECONOMIC THEORY
- EXAMPLE: MODEL OF THE ECONOMY
- USUAL METHODOLOGY: ECONOMETRICS (E.G., MULTIPLE REGRESSION ANALYSIS, TWO-STAGE LEAST SQUARES)

PHYSICAL MODELS

- DYNAMIC CAUSAL MODEL (EQUATIONS OF MOTION)
- PHYSICS
- EXAMPLE: RADAR TRACKING OF A MISSILE
- USUAL METHODOLOGY: KALMAN FILTERING

PURELY STOCHASTIC MODELS (UNIVARIATE, NO EXOGENOUS VARIABLES)

- DESCRIBE STOCHASTIC BEHAVIOR
- TIME SERIES ANALYSIS (EMPIRICAL: NO UNDERLYING ECONOMIC OR PHYSICAL MODEL)
- EXAMPLE: FORECASTING PRODUCT DEMAND
- USUAL METHODOLOGY: BOX-JENKINS (ARIMA) METHODOLOGY

COMBINATION DYNAMIC-STOCHASTIC MODELS

- FEW VARIABLES (BUT MORE THAN ONE)
- SIMPLE (EMPIRICAL) MODEL OF RELATIONSHIP
- EXAMPLES: TRANSFER-FUNCTION MODEL, FUEL-MIXTURE CONTROL
- USUAL METHODOLOGY: BOX-JENKINS; KALMAN FILTERING, “STATE-SPACE” MODELS

FORECASTING ACCURACY

STOCHASTIC VS. HEURISTIC

	FORECAST ERROR VARIANCE									
LEAD TIME	1	2	3	4	5	6	7	8	9	10
MSE (BROWN)	102	158	218	256	363	452	554	669	799	944
MSE(B-J)	42	91	136	180	222	266	317	371	427	483

ECONOMETRIC VS. STOCHASTIC

	THEIL COEFFICIENT	
MODEL	PRICE	QUANTITY
ECONOMETRIC	0.80	0.65
BOX-JENKINS	1.00	0.70
RANDOM WALK	1.00	1.00
MEAN	18.23	0.96

FORECASTING DIFFICULTY

STOCHASTIC MODEL

- CAN INVOLVE A SINGLE VARIABLE
- OPTIMAL FORECAST READILY COMPUTED

ECONOMETRIC MODEL

- DATA REQUIRED FOR ALL MODEL VARIABLES (PAST AND FUTURE)
- FORECASTS FOR ALL MODEL VARIABLES
- FOR OPTIMAL FORECAST, NEED STOCHASTIC MODELS FOR ALL EXOGENOUS VARIABLES

MODEL DEVELOPMENT EFFORT

TECHNICAL SKILLS REQUIRED FOR BOTH ECONOMETRIC AND STOCHASTIC MODELING

NO AUTOMATIC AID FOR ECONOMETRIC MODELING

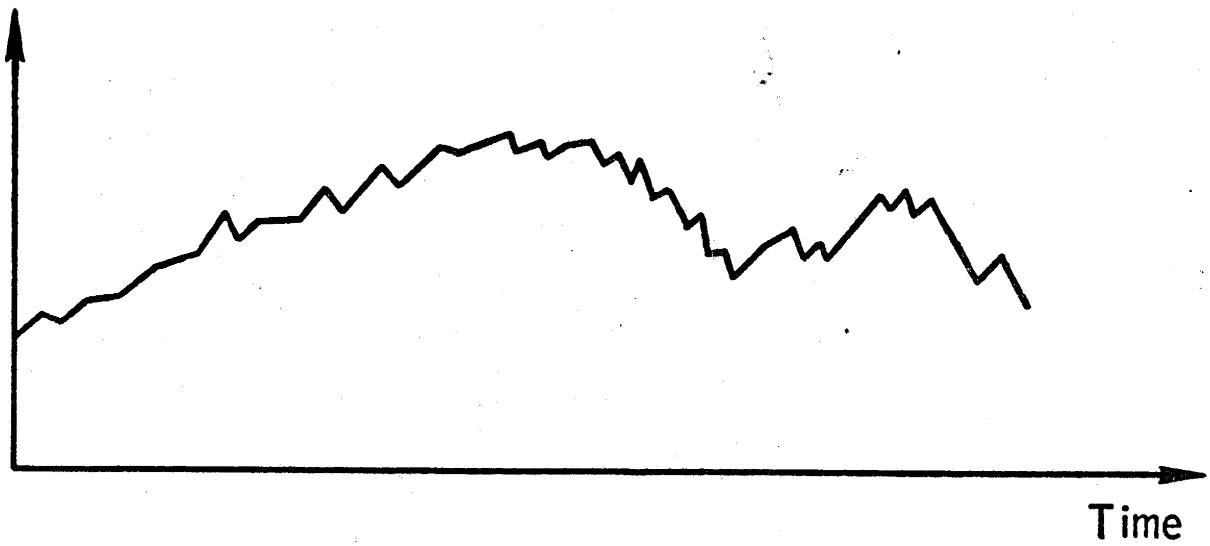
THE BOX-JENKINS METHODOLOGY ENABLES RAPID DEVELOPMENT OF STOCHASTIC MODELS

CAN ALSO ASSIST DEVELOPMENT OF ECONOMETRIC AND CONTROL MODELS

2. TECHNICAL INTRODUCTION TO THE BOX-JENKINS METHODOLOGY

WHAT IS TIME SERIES ANALYSIS?

EXAMPLE OF A TIME SERIES (STOCHASTIC PROCESS):



USES:

- FREQUENCY RESPONSE STUDY (SPECTRAL ANALYSIS)
- FORECASTING
- SIMULATION
- CONTROL

LAST THREE ITEMS REQUIRE MODEL-BUILDING

FORECASTING

INITIAL APPROACHES

- FITTED MODELS (QUICK, NOT OPTIMAL)
- ECONOMETRIC MODELS (EXPENSIVE, NOT APPROPRIATE FOR MOST FORECASTING SITUATIONS)

SUBSEQUENT APPROACHES

- BUILD STOCHASTIC (OR STOCHASTIC-DYNAMIC) MODEL
- DERIVE OPTIMAL FORECASTER
- NEED APPROPRIATE AND FLEXIBLE CLASS OF STOCHASTIC MODELS

KALMAN FILTERING (STATE SPACE): BEST SUITED FOR PHYSICS SITUATIONS, WHERE UNDERLYING PHYSICS IS KNOWN AND IMPORTANT (MANY PARAMETERS, SOMEWHAT COMPLICATED, “OVERKILL” FOR MANY APPLICATIONS)

BOX-JENKINS (ARIMA) MODELS: WIDE APPLICABILITY, EMPIRICAL, RELATIVELY QUICK

- STATIONARY OR NONSTATIONARY
- SEASONAL OR NONSEASONAL
- USED WITH OR WITHOUT ECONOMETRIC MODEL

BOX-JENKINS MODEL

$z_t = \varphi_1 z_{t-1} + \varphi_2 z_{t-2} + \ldots + \varphi_p z_{t-p} + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \ldots - \theta_q a_{t-q}$

WHERE

$z_t, z_{t-1},\ldots$ IS THE OBSERVED TIME SERIES

$\varphi_1, \varphi_2, \ldots, \varphi_p, \theta_1, \theta_2, \ldots, \theta_q$ ARE PARAMETERS

$a_t, a_{t-1},\ldots,$ IS A “WHITE NOISE” SEQUENCE (A SEQUENCE OF UNCORRELATED RANDOM VARIABLES HAVING ZERO MEAN)

OR, IN COMPACT, “OPERATOR,” NOTATION:

$\Phi(B)z_t = \Theta(B)a_t$

WHERE

$Bz_t = z_{t-1}$ (I.E., B DENOTES THE BACKWARD DIFFERENCE OPERATOR)

ITERATIVE PROCEDURE FOR DEVELOPING BOX-JENKINS MODELS

- STATISTICS SUGGEST MODEL STRUCTURE
- PARAMETERS ESTIMATED
- DIAGNOSTIC CHECKING
- IF MODEL INADEQUATE, REPEAT PROCEDURE

PRELIMINARY STATISTICAL ANALYSIS

IDENTIFY DEGREE AND STRUCTURE OF $\Phi(B)$ AND $\Theta(B)$ POLYNOMIALS

TWO USEFUL FUNCTIONS TO ASSIST MODEL IDENTIFICATION

AUTOCORRELATION FUNCTION (ACF):

$\rho_k = \text{corr}(z_t, z_{t-k}) = \text{cov}(z_t, z_{t-k}) / \text{var}(z_t)$

PARTIAL AUTOCORRELATION FUNCTION PACF):

τ_k = k-th COEFFICIENT OF LEAST-SQUARES AUTOREGRESSIVE (AR) MODEL OF ORDER k

ACF “CUTS OFF” AT ORDER q OF PURE MOVING-AVERAGE (MA) PROCESS (p=0)

PACF “CUTS OFF” AT ORDER OF PURE AUTOREGRESSIVE (AR) PROCESS (q=0)

ESTIMATION

PURE AR (NO θs) – LINEAR STATISTICAL MODEL:

$$\underline{z} = Z'\underline{\varphi} + \underline{a}$$

$$\underline{\hat{\varphi}} = (ZZ')^{-1}Z\underline{z}$$

IF θs PRESENT – NONLINEAR STATISTICAL MODEL:

$$a_t = \Theta^{-1}(B)\ \Phi(B)\ z_t$$

I.E.,

$$a_t = a_t(\underline{\varphi}, \underline{\theta}, \underline{z}) = a_t(\underline{\beta}, \underline{z})$$

EXPANDING IN A TAYLOR SERIES AROUND A "GUESS VALUE," $\underline{\beta}_0$:

$$\approx a_t|_{\underline{\beta}=\underline{\beta}0} + \sum_{i=1}^{p+q} (\beta_i - \beta_{io}) \frac{\partial a_t}{\partial \beta_i} |_{\underline{\beta}=\underline{\beta}0}$$

WHICH IS A LINEAR MODEL WITH PARAMETER $\underline{\delta} = \underline{\beta} - \underline{\beta}_0$.

THE PARAMETER ESTIMATES ARE DETERMINED ITERATIVELY (E.G., THE GAUSS-MARQUARDT METHOD OR A NUMERICAL OPTIMIZATION METHOD)

OPTIMAL FORECASTER

THE OPTIMAL FORECASTER MINIMIZES THE MEAN SQUARED ERROR OF PREDICTION

$\hat{z}_t(1)$ = 1-AHEAD FORECAST MADE FROM TIME t

$$\hat{z}_t(1) = \hat{\varphi}_1 z_t + \cdots + \hat{\varphi}_p z_{t-p} - \hat{\theta}_1 \hat{a}_{t-1} - \cdots - \hat{\theta}_q \hat{a}_{t-q}$$

WHERE

$$\hat{a}_t = z_t - \hat{z}_{t-1}(1)$$

SEASONALITY

A REASONABLE MODEL IS

$$\Phi_s(B^s)z_t = \Theta_s(B^s)e_t$$

WHERE e_t IS CORRELATED WITH e_{t-1}, e_{t-2},….

THE MODEL RESIDUALS MAY BE REPRESENTED BY

$$\Phi(B)\, e_t = \Theta(B)a_t$$

WHERE THE at ARE WHITE.

HENCE, COMBINING, THE MODEL IS:

$$\Phi_s(B^s)\, \Phi(B)\, z_t = \Theta_s(B^s)\, \Theta(B)\, a_t$$

EXPONENTIAL SMOOTHING

EXPONENTIAL SMOOTHING IS A SPECIAL CASE OF

$$\Phi(B)\, z_t = \Theta(B) a_t$$

WITH

$$\Phi(B) = 1 - B \;\text{ AND }\; \Theta(B) = 1 - \alpha B ,$$

I.E.,

$$z_t = z_{t-1} + a_t - \alpha a_{t-1} .$$

THE LEAST-SQUARES FORECASTER IS:

$$\hat{z}_t(1) = z_t - \alpha \hat{a}_t$$

OR

$$\hat{z}_t(1) = (1-\alpha)[z_t + \alpha z_{t-1} + \alpha^2 z_{t-2} + \cdots]$$

3. FORECASTING ACCURACY COMPARISONS

CRITERIA FOR FORECASTING PERFORMANCE

FORECAST ERROR VARIANCE, OR MEAN SQUARED ERROR (MSE) OF PREDICTION:

z_t = OBSERVED VALUE AT TIME t

$\hat{z}(\ell) = \ell$ - AHEAD FORECAST MADE FROM TIME t

MSE = $v_\ell = E(z_{t+1} - \hat{z}_t(\ell))^2$

THEIL COEFFICIENT:

$$U = \sqrt{\frac{\sum (z_t - \hat{z}_t)^2}{\sum (z_t - z_{t-1})^2}}$$

<u>BOX-JENKINS VS. BROWN'S METHOD</u>

REF: BOX, JENKINS AND REINSEL, *TIME SERIES ANALYSIS, FORECASTING AND CONTROL*

BROWN'S METHOD:

1. A FORECAST FUNCTION IS SELECTED FROM A GENERAL CLASS OF LINEAR COMBINATIONS AND PRODUCTS OF POLYNOMIALS, EXPONENTIALS, SINES AND COSINES

2. THE SELECTED FORECAST FUNCTIONS ARE FITTED TO DATA BY A "DISCOUNTED LEAST SQUARES" PROCEDURE. MODEL PARAMETERS ARE CHOSEN TO MINIMIZE

$$\sum_{j=0}^{\infty} \beta^j (z_{t-j} - \hat{z}_t(j))^2$$

DATA:

DAILY CLOSING IBM STOCK PRICES, JUNE 29, 1959 – NOVEMBER 2, 1962

BROWN'S MODEL (TRIPLE EXPONENTIAL SMOOTHING):

$$\hat{z}_t(\ell) = C_0(t) + C_1(t)\ell + (1/2)C_2(t)\ell^2$$

WHERE THE C's ARE ADAPTIVE COEFFICIENTS.

BOX-JENKINS MODEL:

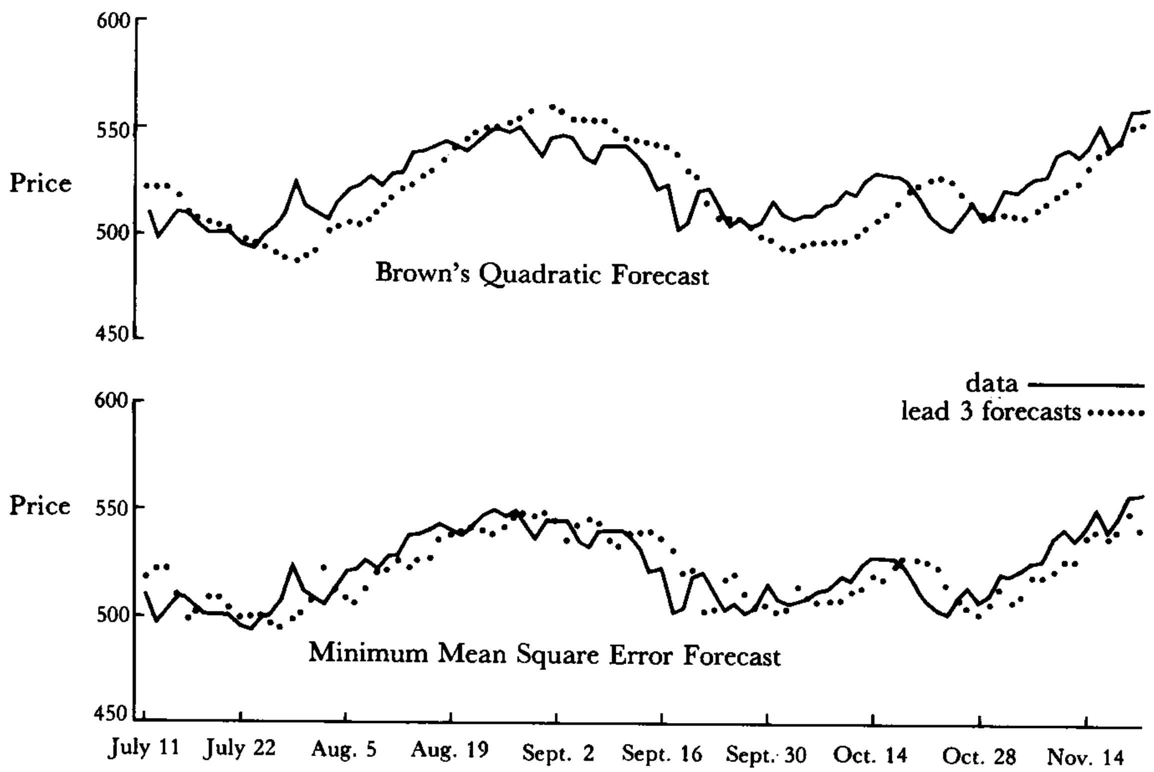
$$\nabla z_t = a_t - .1a_{t-1}$$

MEAN SQUARED FORECAST ERRORS:

LEAD TIME (ℓ)	1	2	3	4	5	6	7	8	9	10
MSE (BROWN)	102	158	218	256	363	452	554	669	799	944
MSE (B-J)	42	91	136	180	222	266	317	371	427	483

BOX-JENKINS VS. BROWN'S METHOD (CONT.)

IBM STOCK PRICE SERIES WITH COMPARISON OF LEAD-3 FORECASTS OBTAINED FROM BEST IMA(0,1,1) PROCESS AND BROWN'S QUADRATIC FORECAST FOR A PERIOD BEGINNING JULY 11, 1960. (FROM BOX, JENKINS, REINSEL REFERENCE)



4. MODEL BUILDING WITH THE BOX-JENKINS METHODOLOGY

MAJOR PHASES OF THE BOX-JENKINS METHODOLOGY

ESTIMATION

- ESTIMATE MODEL PARAMETERS
- ANALYZE MODEL RESIDUALS, REVISE MODEL IF NECESSARY

FORECASTING (USING THE FINISHED MODEL)

- OPTIMAL FORECASTS
- TOLERANCE LIMITS

SIMULATION (USING THE FINISHED MODEL)

- INPUT TO OTHER MODELS (E.G., AN ECONOMIC ANALYSIS OF ALTERNATIVE MODES OF PHARMACEUTICAL MANUFACTURE)
- USED TO TEST MODELS OR DECISION RULES (E.G., TO COMPARE INVENTORY RESTOCKING RULES)

STOCHASTIC MODEL BUILDING

- IDENTIFICATION
- FITTING
- DIAGNOSTIC CHECKING

DIAGNOSTIC CHECKS SUGGEST MODIFICATIONS

MODEL IDENTIFICATION

AUTOCORRELATION FUNCTION (ACF) AND PARTIAL AUTOCORRELATION FUNCTION (PACF)

NEED A STATIONARY VARIATE

HOMOGENEOUS NONSTATIONARITY IS COMMON – ACF DOES NOT DIE OUT

ACHIEVE STATIONARITY BY DIFFERENCING UNTIL THE ACF DIES OUT

HOMOGENEOUS NONSTATIONARITY

$\Phi(B)\, z_t = \Phi'(B)(1 - B)^d z_t$

$= \Phi'(B)\nabla^d z_t$

$= \Phi'(B) w_t$

$\Phi(B)$ HAS d ZEROS (ROOTS) ON THE UNIT CIRCLE – z_t NONSTATIONARY

Φ' HAS ALL ZEROS OUTSIDE THE UNIT CIRCLE – w_t STATIONARY

(RECALL $\nabla z_t = (1-B)z_t = z_t - z_{t-1}$)

SEASONAL HOMOGENEOUS NONSTATIONARITY

IF SEASONAL NONSTATIONARITY IS PRESENT, THE ACF HAS PERIODIC PEAKS THAT DO NOT DIE OUT

APPROACH: TAKE SEASONAL DIFFERENCES:

$$w_t = (1 - B^s)^{d_s} z_t$$

$$= \nabla_s^{d_s} z_t$$

STATIONARY TIME SERIES

ACF CHARACTERIZES (UNIQUELY DEFINES) STATIONARY SERIES

PACF ALSO AIDS IDENTIFICATION

PURE AUTOREGRESSIVE (AR) PROCESS:

$$\Phi(B)w_t = a_t$$

WHERE

$$\Phi(B) = 1 – \varphi_1B - \ldots - \varphi_pB_p \ .$$

PACF CUTS OFF AT ORDER p (ACF TAILS OFF)

PURE MOVING AVERAGE (MA) PROCESS:

$$w_t = \Theta(B)a_t$$

WHERE

$$\Theta(B) = 1 – \theta_1B - \ldots - \theta_qB_q$$

ACF CUTS OFF AT ORDER q (PACF TAILS OFF)

MIXED ARMA PROCESS (ARIMA PROCESS)

$$\Phi(B)\ \nabla^{d} z_t = \Theta(B) a_t$$

AUTOREGRESSIVE-MOVING AVERAGE (ARMA) PROCESS OF ORDER (p, d, q)

USUALLY CALLED AN AUTOREGRESSIVE-INTEGRATED-MOVING-AVERAGE (ARIMA) PROCESS

ACF TAILS OFF AFTER ORDER max (0, q-p)

PACF TAILS OFF AFTER ORDER max (0, p-q)

<u>IDENTIFICATION EXAMPLES</u>

(REF: TABLE 6.1 OF BOX, JENKINS AND REINSEL, *TIME SERIES ANALYSIS, FORECASTING AND CONTROL*)

BEHAVIOR OF THE AUTOCORRELATION FUNCTIONS FOR THE d-th DIFFERENCE OF AN ARIMA PROCESS OF ORDER (p,d,q).

Order	Behavior of ρ_k	Behavior of φ_{kk}	Preliminary estimates from	Admissible Region
(1,d,0)	Decays exponentially	Only φ_{11} nonzero	$\Phi_1 = \rho_1$	$-1 < \varphi_1 < 1$
(0,d,1)	Only ρ_1 nonzero	Exponential dominates decay	$\rho_1 = -\theta_1/(1 + \theta_1^2)$	$-1 < \theta_1 < 1$
(2,d,0)	Mixture of exponentials or damped sine wave	Only φ_{11} and φ_{22} nonzero	$\Phi_1 = \rho_1(1 - \rho_2)/(1 - \rho_1^2)$ $\Phi_2 = (\rho_2 - \rho_1^2)/(1 - \rho_1^2)$	$-1 < \varphi_2 < 1$ $\varphi_2 + \varphi_1 < 1$ $\varphi_2 - \varphi_1 < 1$
(0,d,2)	Only ρ_1 and ρ_2 nonzero	Dominated by mixture of exponentials or damped sine wave	$\rho_1 = -\theta_1(1 - \theta_2)/(1 + \theta_1^2 + \theta_2^2)$ $\rho_2 = -\theta_1/(1 + \theta_1^2 + \theta_2^2)$	$-1 < \theta_2 < 1$ $\theta_2 + \theta_1 < 1$ $\theta_2 - \theta_1 < 1$
(1,d,1)	Decays exponentially from first lag	Dominated by exponential decay from first lag	$\rho_1 = (1 - \theta_1\varphi_1)(\varphi_1 - \theta_1)/(1+ \theta_1^2 - 2\Phi_1\theta_1)$ $\rho_2 = \rho_1\varphi_1$	$-1 < \varphi_1 < 1$ $-1 < \theta_1 < 1$

CAUTION

ESTIMATED AUTOCORRELATIONS MAY BE HIGHLY AUTOCORRELATED, AND MAY HAVE LARGE VARIANCES

USE THE ACF ONLY TO SUGGEST MODELS TO FIT (AMOUNT AND TYPE OF DIFFERENCING, NUMBER OF φs AND θs)

USE LEAST-SQUARES PROCEDURE TO OBTAIN GOOD ESTIMATES FOR THE PARAMETERS

RELY ON DIAGNOSTIC CHECKS TO ACCEPT OR REJECT FITTED MODELS

MODEL FITTING

SPECIFY p, q, AND PRELIMINARY ESTIMATES (GUESS VALUES) OF PARAMETERS

USE AN AVAILABLE COMPUTER PROGRAM (STATISTICAL SOFTWARE PACKAGE) TO ESTIMATE THE PARAMETERS

FOR NONLINEAR MODELS ($q > 0$ OR SEASONAL COMPONENTS), ESTIMATION INVOLVES USE OF AN ITERATIVE ESTIMATION PROCEDURE (E.G., GAUSS-MARQUARDT, GENERAL NONLINEAR ESTIMATION ROUTINE)

DIAGNOSTIC CHECKING

SIGNIFICANCE OF VARIOUS STATISTICS IS COMPUTED FROM THE MODEL "RESIDUALS" (ERROR TERMS):

$$\hat{a}_t = \hat{\Theta}^{-1}(B)\hat{\Phi}(B)z_t$$

MEAN (t-TEST)

PACF (t-TEST ON EACH VALUE)

ACF (t-TEST ON EACH VALUE, χ^2 (CHI-SQUARED) TEST ON ENTIRE FUNCTION

SPECTRUM (GRENANDER-ROSENBLATT TEST)

EXAMPLE OF MODEL MODIFICATION

SUPPOSE THE CORRECT MODEL IS OF ORDER (0,2,2), BUT THAT THE FITTED MODEL IS:

$$\nabla z_t = (1 + .6B)e_t$$

SUPPOSE THAT THE MODEL SUGGESTED FOR THE RESIDUALS (e_t's) IS:

$$\nabla e_t = (1 + .8B)a_t$$

THESE RESULTS SUGGEST THAT AN IMPROVED MODEL WOULD BE:

$$\nabla z_t = (1 - .2B - .48B^2)a_t$$

THIS SUGGESTS A MODEL OF ORDER (0,2,2) SHOULD BE EXAMINED

MODEL SIMPLIFICATION

A MODEL OF THE FORM

$$(1 - \varphi B)(1 - B)z_t = (1 - \theta)a_t$$

MIGHT BE REDUCIBLE TO

$$(1 - \varphi B)z_t = a_t$$

IF θ IS CLOSE TO 1.

5. APPLICATION TO ECONOMETRIC AND CONTROL PROBLEMS

BASIC APPLICATION

PURE STOCHASTIC MODEL:

$$\Phi(B)z_t = \Theta(B)a_t$$

STOCHASTIC-DYNAMIC MODELS

ECONOMETRIC MODELS

PHYSICAL MODELS (E.G., RADAR TRACKING OF A MISSILE)

CONTROL MODELS:

$z_t = L_1^{-1}(B)\, L_2(B)\, B^b\, x_t + \Phi^{-1}(B)\Theta(B)a_t$

$= V(B)x_t + \Phi^{-1}(B)\Theta(B)a_t$

WHERE

x_t IS A STOCHASTIC PROCESS (CONTROL VARIABLE, LEADING INDICATOR); V(B) IS THE IMPULSE RESPONSE FUNCTION OF x_t

IDENTIFICATION OF STOCHASTIC-DYNAMIC MODELS

THE CROSS-CORRELATION FUNCTION (CCF) ASSISTS IDENTIFICATION OF THE TRANSFER FUNCTION:

$$\gamma_{x,z}(k) = ccor(x_t, z_{t+k}) = \frac{\operatorname{cov}(x_t, z_{t+k})}{\sqrt{\operatorname{var}(x_t)\operatorname{var}(z_{t+k})}}$$

THE RELATIONSHIP IS COMPLICATED IF x_t IS NOT “WHITE” (UNCORRELATED, ZERO MEAN)

IF WE “PREWHITEN” x_t, THE RELATIONSHIP IS SIMPLE.

PREWHITENING THE CONTROL (INPUT) VARIABLE

DETERMINE A STOCHASTIC-PROCESS MODEL FOR x_t:

$$x_t = \Phi_x^{-1}(B)\ \Theta_x(B)\ a_{xt}$$

THE PREWHITENED SERIES IS:

$$a_{xt} = \Theta_x^{-1}(B)\ \Phi_x(B)\ x_t$$

THE ORIGINAL MODEL BECOMES:

$$y_t = \Phi_x^{-1}(B)\ \Theta_x(B)\ z_t = L_1^{-1}(B)\ L_2(B)\ B^b\ a_{xt} + \Phi_x^{-1}(B)\ \Theta_x(B)\ \Phi^{-1}(B)\ \Theta(B)\ a_t$$

IDENTIFICATION USING THE PREWHITENED INPUT VARIABLE

THE CROSS-CORRELATION FUNCTION OF (a_{xt}, y_t) IS:

$$\gamma_{ax,y}(k) = V_k \sigma_{ax}^2$$

SO THE TRANSFER FUNCTION V IS DIRECTLY PROPORTIONAL TO THE CCF

HENCE WE CAN DEDUCE A TENTATIVE FORM OF L_1, L_2 FROM:

$$V(B) = L_1^{-1}(B)\, L_2(B)\, B^b$$

OPTIMAL FORECASTER FOR DYNAMIC-STOCHASTIC MODEL

ECONOMETRIC MODEL WITH LEADING INDICATOR x_t:

$$z_t = L_1^{-1}(B)\, L_2(B)\, B^b\, x_t + \Phi^{-1}(B)\, \Theta(B)\, a_t$$

OR

$$L_1(B)\, \Phi(B)\, z_t = \Phi(B)\, L_2(B)\, x_{t-b} + L_1(B)\, \Theta(B)\, a_t$$

OR

$$\Phi^*(B)\, z_t = \Lambda^*(B)\, z_{t-b} + \Theta^*(B)\, a_t$$

THE OPTIMAL FORECASTER FOR THIS MODEL IS:

$$\hat{z}_t(\ell) = \varphi^*{}_1 z'_{t+\ell-1} + \cdots + \varphi^*{}_{p^*} z'_{t+\ell-p^*} + \lambda^*{}_0 x'_{t-b+\ell} - \cdots - \lambda^*{}_{r^*} x'_{t-b+\ell-r^*} + a'_{t+\ell} - \theta^*{}_1 a'_{t+\ell-1} - \cdots - \theta^*{}_{q^*} a'_{t+\ell-q^*}$$

WHERE

$$z'_{t+\ell} = \begin{cases} z_{t+\ell} & \ell \le 0 \\ \hat{z}_t(\ell) & \ell > 0 \end{cases}$$

$$x'_{t+\ell} = \begin{cases} x_{t+\ell} & \ell \le t_0 \\ \hat{x}_t(\ell) & \ell > t_0 \end{cases}$$

$$a'_{t+\ell} = \begin{cases} a_{t+\ell} & \ell \le 0 \\ 0 & \ell > 0 \end{cases}$$

WHERE t0 IS THE POINT IN TIME TO WHICH x_t IS KNOWN, AND THE QUANTITY $\hat{x}_t(\ell)$ IS THE OPTIMAL FORECAST FROM THE STOCHASTIC MODEL FOR x_t .

THUS THE OPTIMAL FORECASTER FOR AN ECONOMETRIC MODEL DEPENDS ON THE OPTIMAL FORECASTER OF THE STOCHASTIC MODEL FOR THE LEADING INDICATOR.

UNLESS WE KNOW xt FOR THE ENTIRE FUTURE PERIOD OVER WHICH WE WISH TO FORECAST, WE MUST USE A STOCHASTIC MODEL TO FORECAST IT IN ORDER TO COMPUTE THE OPTIMAL FORECAST FOR THE ECONOMETRIC MODEL.

SUMMARY

THE BOX-JENKINS APPROACH IS A POWERFUL METHOD FOR DETERMINING MATHEMATICAL MODELS (REPRESENTATIONS) OF A WIDE VARIETY OF STOCHASTIC-PROCESS PHENOMENA.

ALTHOUGH THE METHOD IS RELATIVELY QUICK AND PRODUCES “PARSIMONIOUS” (NOT OVERLY ELABORATE) MODELS, THE COMPUTATIONS REQUIRED TO DEVELOP THE MODEL AND TO DETERMINE OPTIMAL FORECASTS FROM THE MODEL ARE COMPLICATED, AND REQUIRE THE USE OF A STATISTICAL COMPUTER PROGRAM.